J. Fluid Mech. (1982) *vol.* 120, *pp.* 199–217
Printed in Great Britain

Theory for electric charging in turbulent pipe flow

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(Received 29 April 1981 and in revised form 11 December 1981)

Based on somewhat simplified profiles of turbulent eddy diffusivity and mean velocity in turbulent flow, an expression is derived for the convection current in a pipe where electrification occurs at the wall. The expression is in explicit analytic form, and applies for all turbulent Reynolds numbers and all fluid conductivities, from conditions where the Debye length is small compared with the diffusion sublayer (typical aqueous solutions) to conditions where the Debye length is large compared with the sublayer (typical liquid hydrocarbons).

1. Introduction

The term 'charging' in pipe flows refers to the presence of a convection or 'streaming' current

$$I = \int_0^a q v_x \, 2\pi r \, dr, \tag{1}$$

where q is the charge density in the fluid, v_x is the axial velocity component, and a the pipe radius. Virtually all fluids have some electrical conductivity, even if very small, and hence contain some ionized species. The equilibrium condition between the charged species and a bounding wall generally induces a finite charge density in the fluid adjacent to the wall, and hence a convection current will occur whenever flow takes place.

Charging is a matter of practical concern only in flows of very-low-conductivity liquids such as fuels and other liquid hydrocarbons, where high voltages can be built up in improperly grounded pipes, and electrostatic hazards may result (Klinkenberg & van der Minne 1958). Most such flows occur at high Reynolds number, and are therefore turbulent.

A general theoretical description of charging is available for laminar pipe flows (Pribylov & Chernyi 1979), but for turbulent flows the situation is not so well in hand. Contributions have been made by Cooper (1953), Klinkenberg (1959, 1964) and Koszman & Gavis (1962*a*). Cooper's was a simple step: he adapted Helmholtz's (1879) classical streaming current equation to turbulent flows by inserting into it Blasius's correlation for turbulent shear stress. This gave him an expression for the high-conductivity limit, where the Debye length is small compared with the diffusion-sublayer thickness (laminar flow and a linear velocity profile being underlying assumptions in Helmholtz's expression). Cooper's formula is useful for typical aqueous

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electrolytes where the Debye length is measured in 10^{-8} m, but it often does not apply to liquid hydrocarbons, where typical conductivities are so low that the Debye length is larger than the diffusion sublayer.

Klinkenberg and Koszman & Gavis addressed the lower-conductivity limit. They argued that, when the Debye layer is thick compared with the diffusion sublayer, the charge is distributed approximately uniformly in the turbulent core, and a Nernst-layer approach can be used to describe the diffusion sublayer. Their expression for the charging current is in approximate agreement with much, though by no means all, of the experimental data for turbulent liquid hydrocarbon flows (Koszman & Gavis 1962*b*; Goodfellow & Graydon 1968; Gibson & Lloyd 1970*a*, *b*; Schwing 1970).

It is worth noting that a quite different turbulent-flow theory had been put forward earlier by Boumans (1957), and taken up recently by Touchard (1978). These authors did not, however, solve for the charge-density profile from basic principles. Rather, they obtained a charge profile for turbulent flow by making some *ad hoc* postulates which, although stated differently, are equivalent to the assumptions that (i) the charge density at the wall, (ii) the charge density gradient at the wall, *and* (iii) the total charge integrated over the pipe cross-section are all properties of the pipe and fluid, and independent of flow conditions. The resulting charge density distribution contains a discontinuity and is physically unrealistic. It is also totally inconsistent with the model presented here, which is based on the equations that govern the charge density in turbulent flows.

Another flawed turbulent-flow theory is the 'high-conductivity' model proposed – rather tentatively, to be sure – by Koszman & Gavis (1962*a*) for conditions where the charged region extends far outside the diffusion sublayer, but still not too far into the central region of the pipe flow. Koszman & Gavis derived their equation for this limit on the assumption that turbulence does not affect the diffusivity outside the diffusion sublayer, which is inconsistent with the definition of the diffusion sublayer.

The purpose of the present paper is to derive a general solution for charging in turbulent pipe flows, based on the equations which govern the charge density. The result will be obtained in analytic form, and is applicable to all flow speeds and fluid conductivities. The solutions of Cooper and Klinkenberg/Koszman & Gavis appear as special limiting cases.

Our one basic assumption, apart from presuming fully turbulent flow conditions and high Schmidt number, is that the charge density q in the fluid is small, in the sense that it arises from such a small imbalance of the positive and negative ions that

$$\frac{|q|}{Fc_0} \ll 1, \tag{2}$$

where F is Faraday's constant and c_0 denotes the concentration of either the positive or negative ions in the fluid under electrically neutral conditions and is related to the fluid's electrical conductivity σ by

$$c_0 = \frac{\sigma}{2F\kappa}, \tag{3}$$

$\kappa = \frac{1}{2}(\kappa_+ + \kappa_-)$ being the average of the mobilities of the positive and negative ions in the fluid. When (2) applies, σ is uniform in the fluid, independent of q. The small-charge-density assumption, which also underlies most previous work in this area, allows the formulation of the theory in terms of the conservation equation for charge density only, without reference to the conservation equations for the particular ionized species that give rise to the charge. At high charge densities, on the other hand, the conservation equations of the individual ionized species must be brought into the theory, which means that one must define the ionization and recombination reactions and their rate coefficients, or, at the very least, the equilibrium constant. None of this information is available for typical hydrocarbon liquids, nor is this kind of data likely to become available readily, considering that the dominant ionized species in liquid hydrocarbons are often unknown impurities.

2. Equations and boundary conditions

Since the velocity distribution is well known in pipe flows, we need only deal here with the equations for the charge distribution in order to evaluate the convective current of (1). The governing equations are the equation of charge conservation

$$\frac{\partial q}{\partial t}+\nabla . \mathbf{j} = 0, \tag{4}$$

the current density $\mathbf{j}$ being given in the small-charge-density approximation by (Huber & Sonin 1977)

$$\mathbf{j} = q\mathbf{v}-D\nabla q-\sigma\nabla\phi, \tag{5}$$

Poisson's equation for the electric field

$$\nabla^2\phi = -q/\epsilon, \tag{6}$$

and the equations that govern the fluid's velocity field. Here $\mathbf{v}$ is the bulk flow velocity, $D = \frac{1}{2}(D_+ + D_-)$ is the average molecular diffusion coefficient (D_+ and D_- being the diffusion coefficients of the positive and negative ions respectively), ϕ is the electric potential, and ϵ is the fluid's permittivity. For an incompressible flow, (4)–(6) yield the charge-relaxation equation

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q-\nabla .(D\nabla q)+\sigma q/\epsilon = 0. \tag{7}$$

In *turbulent flows* one must deal with ensemble-averaged versions of these equations. Procedures for ensemble averaging are well known. If we confine our attention to (i) a steady, turbulent pipe flow with (ii) fully developed mean velocity profile – but not necessarily a fully developed concentration profile – under conditions where (iii) axial diffusion of charge is negligible compared with radial diffusion, the ensemble-averaged version of (7) is

$$\bar{v}_x\frac{\partial \bar{q}}{\partial x}-\frac{1}{r}\frac{\partial}{\partial r}\left[r(D+D_{\mathrm{T}})\frac{\partial \bar{q}}{\partial r}\right]+\frac{\sigma\bar{q}}{\epsilon} = 0. \tag{8}$$

Here, $\bar{v}_x$ is the local ensemble-averaged velocity, $\bar{q}$ is the local ensemble-averaged charge density, and we have made the usual assumption that the turbulent contribution $\overline{v_r'q'}$ to the radial current density is given by

$$\overline{v_r'q'} = -D_{\mathrm{T}}\frac{\partial \bar{q}}{\partial r}. \tag{9}$$

The primed quantities refer to the fluctuating components, and D_T is a turbulent-diffusion coefficient. The contribution to the current density of the turbulent fluctuations in conductivity can be shown to be negligible as long as the low-charge-density assumption (2) applies. Consistent with our assumptions, the averaged radial and axial current densities are given by

$$\bar{j}_r = -(D+D_T)\frac{\partial \bar{q}}{\partial r} - \sigma\frac{\partial \bar{\phi}}{\partial r}, \tag{10}$$

$$\bar{j}_x = \bar{q}\bar{v}_x - \sigma\frac{\partial \bar{\phi}}{\partial x}. \tag{11}$$

The averaged version of Poisson's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}}{\partial r}\right) + \frac{\partial^2 \bar{\phi}}{\partial^2 x} = -\frac{\bar{q}}{\epsilon}. \tag{12}$$

Equation (8) requires boundary conditions on $\bar{q}$ at the pipe inlet and at the pipe walls. At the inlet $\bar{q}$ is specified as having whatever value is appropriate for the source from which the fluid issues. For example if the fluid enters the pipe from an uncharged reservoir we set

$$\bar{q} = 0 \quad \text{at} \quad x = 0. \tag{13}$$

The boundary condition on the wall is less straightforward. Little is known about the electrification process at boundaries of liquid hydrocarbons, for which the present analysis is mainly intended. It suffices to assume, however, that a charge density tends to be induced in the fluid adjacent to the wall for some reason. For example, one of the ionic species in the liquid may suffer an oxidation–reduction reaction at the wall interface. Suppose q_w is the charge density that would exist in the fluid next to the wall under equilibrium conditions, when all net fluxes – and the net current – there are zero. When the charge density q at the wall is not equal to q_w, a current will pass between the fluid and the wall. In the absence of specific data, we postulate a linear current–charge relation

$$j_r = \frac{j_w}{|q_w|}[q - q_w] \quad \text{at} \quad r = a, \tag{14}$$

which contains two coefficients q_w and j_w. We consider these to be empirical properties of the fluid/wall combination and the ionic composition of the fluid. q_w is the charge density that exists in the fluid next to the wall when $j_r = 0$; j_w is the magnitude of the current that passes between the fluid and the wall if $q = 0$ at the wall. The direction of the current is from the wall to the fluid if $q_w > 0$, and from the fluid to the wall if $q_w < 0$. Since (14) is linear, it applies to $\bar{j}$ and $\bar{q}$ as well as to j and q.

For an axisymmetric pipe (14) must be complemented by the symmetry condition of zero radial charge density gradient on the pipe axis.

The ensemble-averaged equations are completed by specifying the average velocity profile $\bar{v}_x(r)$ and the eddy-diffusivity profile $D_T(r)$ for the turbulent pipe flow. For smooth-walled pipes, quite accurate semi-empirical expressions are available. Spalding (1961), for example, gives an implicit equation for $\bar{v}_x$. Reichardt (1951) gives an expression for D_T which Petukhov (1970) has found to be very adequate for describing mass-transfer processes at high Schmidt numbers (up to 10^5), thus confirming its accuracy inside the diffusion sublayer, and which Quarmby & Anand (1969) have verified in the regions outside the diffusion sublayer.

In the present paper we take a more approximate approach in order to derive an analytic solution. The analysis of § 3 is based on the simplified eddy-diffusivity profile

$$D_{\mathrm{T}}\begin{cases} \ll D & (0 < y < \delta), \\ = \chi v_* y & (\delta \leqslant y \leqslant \tfrac{1}{4}a), \\ = \tfrac{1}{4}\chi v_* a & (\tfrac{1}{4}a < y < a). \end{cases} \tag{15}$$

Here $y = a - r$ is the distance measured out from the wall,

$$v_* \equiv (\tau_{\mathrm{W}}/\rho)^{\frac{1}{2}} \tag{16}$$

is the friction velocity, τ_{W} being the (averaged) shear stress at the wall and ρ the fluid density, and

$$\delta = \frac{k}{S^{1/m}}\frac{\nu}{v_*}, \tag{17}$$

is the effective diffusion-sublayer thickness, ν being the fluid's kinematic viscosity and $S \equiv \nu/D$ the Schmidt number. χ is von Kármán's constant,

$$\chi = 0{\cdot}4, \tag{18a}$$

and k and m are empirical coefficients, for which we suggest the following values as best estimates for dealing with high-Schmidt-number flows:

$$k = 11{\cdot}7, \quad m = 3. \tag{18b}$$

Equation (15) takes account of the major accepted features of the eddy-diffusivity profile: D_{T} is small compared with D very near the wall, where yv_*/ν is sufficiently small, but increases rapidly with y (as y^3, according to Reichardt) until it rather abruptly overwhelms D as y reaches a value of about δ, δ being still very small compared with the pipe radius a. Thereafter it increases linearly with y until y becomes measurable compared with the pipe radius a (about $0{\cdot}2a$, say), after which it levels off to an approximately constant value in the core region of the flow. The value of D_{T} in the core region scales with the friction velocity as indicated. The values of k and m in (18b) are obtained by making the identification $\delta/2a \equiv N^{-1}$, where N is the diffusion Nusselt number based on pipe diameter, and using Petukhov's high-Schmidt-number correlation for N. Petukhov's high-Schmidt-number correlation is identical with the correlation of Reichardt in the same limit, and agrees very well with experiment. (It differs somewhat from Deissler's (1955) form, which would imply $k = 8{\cdot}96$ and $m = 4$.) Outside the diffusion sublayer our profile fits experimental data (Laufer 1954; Quarmby & Anand 1969) in the linear region, but overestimates the diffusivity somewhat in the uniform-diffusivity core of the flow. The latter has no perceptible effect on our prediction of a streaming current, since at high Schmidt numbers it turns out that only the eddy diffusivity near the wall is important.

For the velocity profile we use the approximation

$$\bar{v}_x = \begin{cases} \dfrac{\tau_{\mathrm{W}}}{\mu}y & (y < \delta), \\ V & (\delta \leqslant y \leqslant a), \end{cases} \tag{19}$$

where V is the superficial velocity in the pipe, and μ is the fluid viscosity. Equation (19) is accurate in the region inside the diffusion sublayer, where, it turns out, accuracy is needed. We shall see that accuracy in $\bar{v}_x$ is not paramount outside δ, provided the volume flow rate is correct, as is the case in (19).

The shear stress τ_W in (16) and (19) can be expressed in terms of the superficial velocity V by means of several available friction-factor correlations. The simplest is the well-known one of Blasius,

$$\tau_W/\rho V^2 = 0{\cdot}0396 R^{-\frac{1}{4}}, \tag{20}$$

where

$$R \equiv \rho V 2a/\mu \tag{21}$$

is the Reynolds number. Blasius's correlation is accurate only for $R < 10^5$. More universal, but less simple, correlations are available, such as the widely used one of Prandtl (Schlichting 1979).

3. Analytical solutions for infinite pipes

3.1. *Equations*

By 'infinite' pipes we mean pipes that are so long that both the charge density and the radial electric field have reached distributions that no longer change with x. (We do not, however, rule out the presence of a uniform axial electric field $\bar{E}_x = -\partial\bar{\phi}/\partial x$.) The governing equations (8) and (12) reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r(D+D_T)\frac{\partial \bar{q}}{\partial r}\right] = \frac{\sigma \bar{q}}{\epsilon}, \tag{22}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}}{\partial r}\right) = -\frac{\bar{q}}{\epsilon}. \tag{23}$$

$\bar{q}$ can be independent of x only if the radial current density at the wall is zero,

$$\bar{j}_r = 0 \quad \text{at} \quad r = a, \tag{24}$$

and it follows that the boundary condition (14) reduces to

$$\bar{q} = q_W \quad \text{at} \quad r = a. \tag{25}$$

Equation (24) implies that at the wall the radial diffusion current, which arises from a charge density gradient ($\bar{q}$ being equal to q_W at the wall, but tending to relax away from the wall), is exactly counterbalanced by a migration current so as to make the net current zero. Using (10) and (23), this statement can also be written as

$$D\frac{\partial \bar{q}}{\partial r}\bigg|_{r=a} = \frac{\sigma}{\epsilon a}\int_0^a \bar{q} r\, dr. \tag{26}$$

Alternatively, (26) can be viewed as being an integral of (22), the condition of zero radial current being implicit in the assumption of an 'infinite' pipe.

3.2. *Previous analytical solutions: two limiting cases*

Two analytical solutions have been previously identified: the classical Helmholtz solution for high-conductivity fluids, adapted to turbulent flow by Cooper, which applies at the high conductivities typical of aqueous electrolytes, and the solution of Klinkenberg (1959, 1964) and Koszman & Gavis (1962*a*), which we shall call the diffusion-layer-dominated solution, and which applies under special conditions more

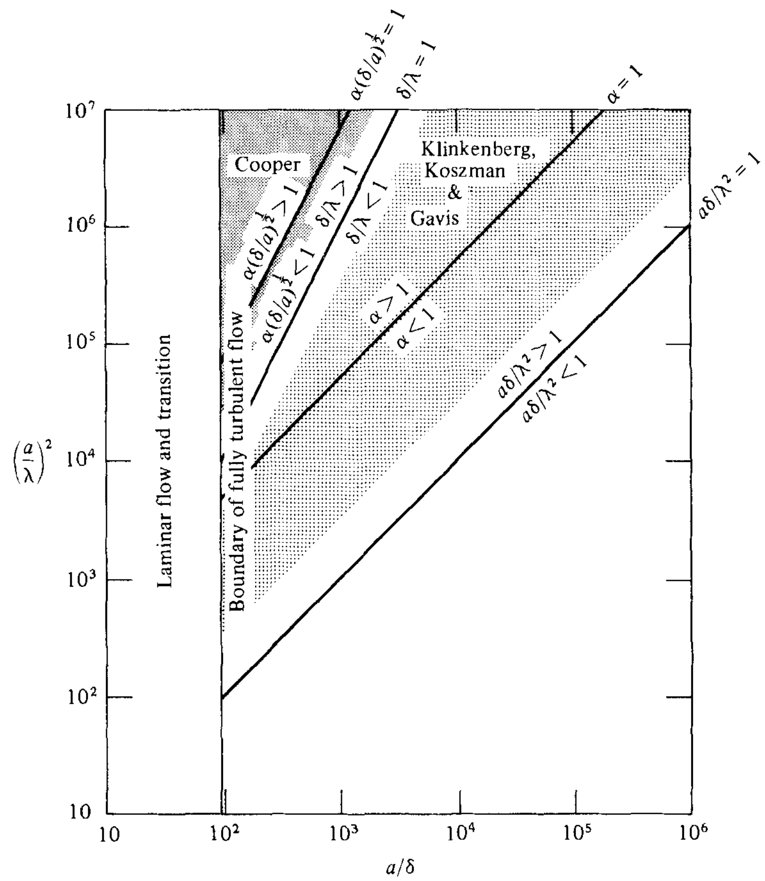

FIGURE 1. Characterization of operating conditions as a point on the $(a^2/\lambda^2,\, a/\delta)$-plane, showing region where the solutions of Cooper and Klinkenberg/Koszman–Gavis are applicable. Assumptions are $\chi = 0{\cdot}4$, $k = 11{\cdot}7$, $m = 3$, $S = 500$.

typical of very-low-conductivity liquids. Each of these solutions, we shall see, applies in a certain limit of the parameters λ^2/δ^2 and a/δ, where

$$\lambda = (\epsilon D/\sigma)^{\frac{1}{2}} \tag{27}$$

is the Debye length, and δ is the diffusion-sublayer thickness.

We note first that on a map of λ^2/δ^2 *vs.* a/δ (figure 1) turbulent-flow solutions apply only for a/δ larger than a minimum value corresponding to the minimum Reynolds number at which fully turbulent flow occurs. The latter is about 4000. Using Blasius's empirical formula and (17) and (18), we obtain

$$\left(\frac{a}{\delta}\right)_{\text{min, turb}} \simeq 12S^{\frac{1}{3}}, \tag{28}$$

which shows that a/δ is always large in turbulent flow, particularly when $S \gg 1$.

Debye-layer-dominant solution $(\lambda^2/\delta^2 \ll 1)$. The diffusion layer δ measures the thickness of the laminar sublayer near the wall, and the Debye length measures the distance to which the charged layer penetrates into the fluid under laminar conditions.

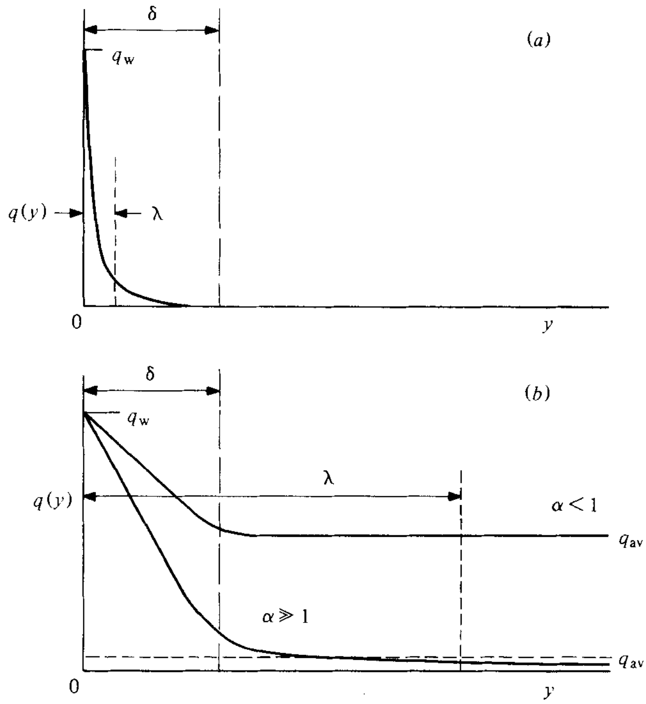

FIGURE 2. Sketch of charge distribution near wall for (a) $\lambda^2 \ll \delta^2$, (b) $\lambda^2 \gg \delta^2$.

If $\lambda^2/\delta^2 \ll 1$, the entire charged layer (the diffuse double layer) is confined to a region very near the wall where turbulent diffusion is negligible and where the mean velocity increases linearly from zero (see figure 2*a*). Helmholtz (1879) showed that when these conditions apply, and when $\lambda \ll a$, as is the case when $\delta \ll a$ and $\lambda < \delta$, the convective current in an infinite pipe is given by

$$I_\infty = -\frac{2\pi a}{\mu}\tau_W \epsilon\zeta, \tag{29}$$

where ζ is the zeta-potential of the wall, or $\phi(a)$, related to the charge density q_W at the wall by

$$\zeta = -\frac{Dq_W}{\sigma}. \tag{30}$$

Equation (30) follows from (10) and (2), with $D_T \ll D$. Equation (29) can thus be expressed as

$$\frac{I_\infty}{q_W \pi a^2 V} = \frac{\tau_W}{\rho V^2} R \left(\frac{\lambda}{a}\right)^2. \tag{31}$$

If one accepts Blasius's correlation, (20), for τ_W, (31) can be expressed as

$$\frac{I_\infty}{q_W \pi a^2 V} = 0{\cdot}0306 \left(\frac{\lambda}{a}\right)^2 R^{\frac{3}{4}}. \tag{32}$$

This simple adaptation of Helmholtz's equation to turbulent flow in the small-Debye-length limit $\lambda^2/\delta^2 \ll 1$ was first derived by Cooper.

Diffusion-layer-dominated solution ($\lambda^2/a\delta \ll 1$, $\lambda^2/a\delta \gg 1$). When the Debye length is large compared with the diffusion layer ($\lambda^2/\delta^2 \gg 1$), the charged region protrudes into the turbulent-flow region outside the diffusion layer as indicated in figure 2(*b*). What is more, because the turbulent diffusivity outside the diffusion layer is so much higher than the diffusion coefficient inside the layer, the charge gradients in the turbulent region will be much lower than those in the diffusion layer. An approximate solution for the convection current can be obtained quickly as follows. Let

$$q_{\text{av}} \equiv \frac{1}{\pi a^2}\int_0^a \bar{q} 2\pi r\, dr, \tag{33}$$

be the average charge density in the flow (figure 2*b*). We make two assumptions. First, that the average charge density is low compared with that of the wall,

$$q_{\text{av}} \ll q_{\text{w}}. \tag{34}$$

Secondly, that, although q_{av} is low, the bulk of the integrated charge is in fact spread out in the turbulent core region rather than inside the higher-charge-density diffusion layers,

$$q_{\text{w}}\, 2\pi a\delta \ll \pi a^2 q_{\text{av}}. \tag{35}$$

These assumptions allow the flux-balance condition at the wall (26) to be written as

$$\frac{Dq_{\text{w}}}{\delta} \simeq \frac{\sigma a q_{\text{av}}}{2\epsilon}, \tag{36}$$

or

$$q_{\text{av}} = \frac{2\epsilon D q_{\text{w}}}{\sigma a\delta} = \frac{2\lambda^2}{a\delta} q_{\text{w}}, \tag{37}$$

and, since the bulk of the charge is distributed in the core where the mean axial speed is quite uniform, the convection current

$$I = \int_0^a \bar{v}_x \bar{q}\, 2\pi r\, dr \tag{38}$$

is to a good approximation given by $q_{\text{av}}\pi a^2 V$, which leads to

$$\frac{I_\infty}{q_{\text{w}}\pi a^2 V} \simeq 2\frac{\lambda^2}{a\delta}. \tag{39}$$

This is in essence the solution of Klinkenberg and Kozman & Gavis, except that their derivation, as well as their treatment of the wall boundary condition, was somewhat more restrictive (specific). Two conditions must be satisfied for this solution to be valid. The first is (34), which is equivalent to $\lambda^2/a\delta \ll 1$. The second is (35), which is equivalent to $\lambda^2/\delta^2 \gg 1$. Figure 1 shows the region of validity on a map of λ^2/δ^2 *vs.* a/δ.

If we substitute for δ from (17), and use Blasius's correlation for τ_{w}, (39) takes the form

$$\frac{I_\infty}{q_{\text{w}}\pi a^2 V} = \frac{0{\cdot}199}{k}\left(\frac{\lambda}{a}\right)^2 R^{\frac{7}{8}} S^{1/m}. \tag{40}$$

Equation (40) can be reduced to Klinkenberg's and Koszman & Gavis's result if one makes an appropriate identification of q_{w} and sets $k = 8{\cdot}96$ and $m = 4$, consistent with Deissler's correlation for diffusion in pipe flows. We prefer the values $k = 11{\cdot}7$

and $m = 3$, which Petukhov has shown to represent diffusion data better at high Schmidt numbers. With the latter values we have

$$\frac{I_\infty}{q_{\mathrm{w}}\pi a^2 V} = 0{\cdot}0170\left(\frac{\lambda}{a}\right)^2 R^{\frac{7}{8}} S^{\frac{1}{3}}. \tag{41}$$

Koszman & Gavis recognized, after comparing their theory with experiment, that their solution was subject to the assumption $\lambda > \delta$, but not that there was an additional assumption implied, $\lambda^2/a\delta \ll 1$. Nor did they – or Klinkenberg – realize that the charge distribution in the pipe does *not* have to be uniform outside the diffusion sublayer for (40) to apply.

3.3. *Present analytical solution*

Using the approximate equations (15) and (19) for D_{T} and $\bar{v}_x$, and the additional simplification of treating the problem as being plane in the region $0{\cdot}75a < r < a$, it is possible to derive an explicit analytic solution which is valid for all turbulent flow conditions.

Using (17) and (15), we cast (22) into the forms

$$\left.\begin{aligned}
&\frac{\partial \tilde{q}_{\mathrm{I}}}{\partial \tilde{y}^2} = \left(\frac{a}{\lambda}\right)^2 \tilde{q}_{\mathrm{I}} \quad \left(0 \leqslant \tilde{y} \leqslant \frac{\delta}{a}\right) \quad \text{(region I)},\\
&\frac{\partial}{\partial \tilde{y}}\left(\tilde{y}\frac{\partial \tilde{q}_{\mathrm{II}}}{\partial \tilde{y}}\right) = \tfrac{1}{4}\alpha^2 \tilde{q}_{\mathrm{II}} \quad \left(\frac{\delta}{a} \leqslant \tilde{y} \leqslant 0{\cdot}25\right) \quad \text{(region II)},\\
&\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{q}_{\mathrm{III}}}{\partial \tilde{r}}\right) = \alpha^2 \tilde{q}_{\mathrm{III}} \quad (0 \leqslant \tilde{r} \leqslant 0{\cdot}75) \quad \text{(region III)},
\end{aligned}\right\} \tag{42}$$

where

$$\tilde{q} \equiv \bar{q}/q_{\mathrm{w}}, \quad \tilde{y} \equiv y/a, \quad \tilde{r} \equiv r/a = 1 - y/a, \tag{43}$$

and

$$\alpha^2 \equiv \frac{4}{k\chi S^{1-1/m}}\frac{a\delta}{\lambda^2}, \tag{44}$$

k, χ and m being the empirical coefficients in (15). Note that the parameter

$$\alpha = a/\lambda_{\mathrm{T}} \tag{45}$$

is the ratio of the pipe radius a to the Debye length λ_{T} based on the *turbulent* eddy diffusivity in the core of the flow,

$$\lambda_{\mathrm{T}} \equiv \left(\frac{\epsilon\chi v_* a}{4\sigma}\right)^{\frac{1}{2}}. \tag{46}$$

Equation (45) follows from (44) and (17).

The solutions to (42) are

$$\left.\begin{aligned}
\tilde{q}_{\mathrm{I}} &= C_1\cosh\left(\frac{a}{\lambda}\tilde{y}\right) + C_2\sinh\left(\frac{a}{\lambda}\tilde{y}\right),\\
\tilde{q}_{\mathrm{II}} &= C_3 I_0(\alpha\tilde{y}^{\frac{1}{2}}) + C_4 K_0(\alpha\tilde{y}^{\frac{1}{2}}),\\
\tilde{q}_{\mathrm{III}} &= C_5 I_0(\alpha\tilde{r}) + C_6 K_0(\alpha\tilde{r}),
\end{aligned}\right\} \tag{47}$$

where I_n and K_n are the nth-order modified Bessel functions of the first and second kind respectively.

The integration constants $C_1, \ldots, C_6$ are evaluated by applying the two general boundary conditions

$$\left.\begin{aligned} \tilde{q}_{\mathrm{I}} &= 1 \quad \text{at} \quad \tilde{y} = 0, \\ \frac{\partial \tilde{q}_{\mathrm{III}}}{\partial \tilde{r}} &= 0 \quad \text{at} \quad r = 0, \end{aligned}\right\} \tag{48}$$

the conditions that the charge density be continuous at the two region interfaces,

$$\left.\begin{aligned} \tilde{q}_{\mathrm{I}} &= \tilde{q}_{\mathrm{II}} \quad \text{at} \quad \tilde{y} = \delta/a, \\ \tilde{q}_{\mathrm{II}} &= \tilde{q}_{\mathrm{III}} \quad \text{at} \quad \tilde{y} = 0{\cdot}25, \end{aligned}\right\} \tag{49}$$

and the conditions that the radial electric field be continuous at the interfaces. Using Gauss's law, and (10) with $\bar{j}_r = 0$ to eliminate the electric field, the continuity conditions for the radial field can be expressed in the form of integral constraints:

$$\left.\begin{aligned} -\frac{\lambda^2}{a^2}\frac{\partial \tilde{q}_{\mathrm{I}}}{\partial \tilde{y}}\bigg|_{y=\delta/a} &= \int_0^{0\cdot 75} \tilde{q}_{\mathrm{III}}\, \tilde{r}\, d\tilde{r} + \int_{\delta/a}^{0\cdot 25} \tilde{q}_{\mathrm{II}}\, d\tilde{y}, \\ -\frac{1}{\alpha^2}\frac{\partial \tilde{q}_{\mathrm{II}}}{\partial \tilde{y}}\bigg|_{\tilde{y}=0\cdot 25} &= \frac{1}{0{\cdot}75}\int_0^{0\cdot 75} \tilde{q}_{\mathrm{III}}\, \tilde{r}\, d\tilde{r}. \end{aligned}\right\} \tag{50}$$

After some algebra, we obtain the following expressions for $C_1, \ldots, C_6$:

$$\left.\begin{aligned} &C_1 = 1, \quad C_2 = \frac{F(\alpha, \delta/\lambda)\cosh(\delta/\lambda) - G(\alpha, \delta/\lambda)\sinh(\delta/\lambda)}{H(\alpha, \delta/\lambda)}, \\ &C_3 = \frac{f(\alpha)}{H(\alpha, \delta/\lambda)}, \quad C_4 = \frac{g(\alpha)}{H(\alpha, \delta/\lambda)}, \\ &C_5 = \frac{I_0(\frac{1}{2}\alpha)}{I_0(\frac{3}{4}\alpha)} C_3 + \frac{K_0(\frac{1}{2}\alpha)}{I_0(\frac{3}{4}\alpha)} C_4, \quad C_6 = 0. \end{aligned}\right\} \tag{51}$$

Here

$$\left.\begin{aligned} f(\alpha) &= I_0(\tfrac{3}{4}\alpha)\, K_1(\tfrac{1}{2}\alpha) - I_1(\tfrac{3}{4}\alpha)\, K_0(\tfrac{1}{2}\alpha), \\ g(\alpha) &= I_0(\tfrac{3}{4}\alpha)\, I_1(\tfrac{1}{2}\alpha) + I_1(\tfrac{3}{4}\alpha)\, I_0(\tfrac{1}{2}\alpha), \\ F(\alpha, \delta/\lambda) &= 2(\tfrac{1}{4}k\chi S^{1-1/m})^{\frac{1}{2}}\left[I_1\left(\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}}\right) f(\alpha) - K_1\left(\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}}\right) g(\alpha)\right], \\ G(\alpha, \delta/\lambda) &= I_0\left(\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}}\right) f(\alpha) + K_0\left(\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}}\right) g(\alpha), \\ H(\alpha, \delta/\lambda) &= G(\alpha, \delta/\lambda)\cosh(\delta/\lambda) - F(\alpha, \delta/\lambda)\sinh(\alpha/\lambda), \end{aligned}\right\} \tag{52}$$

and I_n and K_n are the nth-order modified Bessel functions of the first and second kind.

Equation (47), with the coefficients as given in (51) and (52), defines the charge distribution in the pipe under general turbulent-flow conditions.

The convection current

$$I \equiv \int_0^a \bar{q}\, \bar{v}_x\, 2\pi r\, dr, \tag{53}$$

can now be evaluated by piecewise integration over the three regions I–III, using the approximate velocity profile of (19):

$$\frac{I_\infty}{q_{\mathrm{w}}\pi a^2 V} = R\frac{\tau_{\mathrm{w}}}{\rho V^2}\int_0^{\delta/a} \tilde{q}_{\mathrm{I}}\, \tilde{y}\, d\tilde{y} + 2\int_{\delta/a}^{0\cdot 25} \tilde{q}_{\mathrm{II}}\, d\tilde{y} + 2\int_0^{0\cdot 75} \tilde{q}_{\mathrm{III}}\, \tilde{r}\, d\tilde{r}. \tag{54}$$

With the appropriate substitutions, and using the fact that $\delta/a \ll 1$, one obtains

$$\frac{I_\infty}{q_w \pi a^2 V} = R\frac{\tau_w}{\rho V^2}\frac{\lambda^2}{a^2}[((\delta/\lambda)\sinh(\delta/\lambda) - \cosh(\delta/\lambda) + 1) + C_2((\delta/\lambda)\cosh(\delta/\lambda) - \sinh(\delta/\lambda))] + \frac{2C_3}{\alpha}I_1(\tfrac{1}{2}\alpha) - \frac{2C_4}{\alpha}\left[K_1(\tfrac{1}{2}\alpha) - 2\left(\frac{\delta}{a}\right)^{\frac{1}{2}}K_1\left(\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}}\right)\right] + \frac{3C_5}{2\alpha}I_1(\tfrac{3}{4}\alpha). \tag{55}$$

Now $\tau_w/\rho V^2$ is a function of R only, and a/δ is therefore a function of only R and S. Equation (55) therefore expresses the dimensionless convection current as a function of $(a/\lambda)^2$, a/δ and S. This solution can be reduced to a simpler, explicit form by considering certain limits in the parameters $(a/\lambda)^2$ and a/δ.

High-α limit. The limit $\alpha \gg 1$ corresponds to conditions where the Debye length based on the turbulent core diffusivity is small compared with the pipe radius, and as a consequence, the charge density is non-zero only in a region close to the wall. Whether that region is smaller or larger than the diffusion layer is left open. Expanding the Bessel functions for large α and keeping only leading terms, we obtain from (51) and (52)

$$\left.\begin{aligned} C_2 &\simeq -\frac{\cosh(\delta/\lambda)}{\sinh(\delta/\lambda)}, \quad C_3 \to 0, \\ C_4 &\simeq \frac{1}{(k\chi S^{1-1/m})^{\frac{1}{2}}K_1(\alpha(\delta/a)^{\frac{1}{2}}\sinh(\delta/\lambda)}, \quad C_5 \to 0. \end{aligned}\right\} \tag{56}$$

Strictly speaking, the above expression for C_2 is derived for $\alpha \gg 1$ and $\alpha(\delta/a)^{\frac{1}{2}} \ll 1$. However,

$$\alpha\left(\frac{\delta}{a}\right)^{\frac{1}{2}} = \frac{2}{(k\chi S^{1-1/m})^{\frac{1}{2}}}\frac{\delta}{\lambda}, \tag{57}$$

becomes comparable to unity only when δ/λ is *very* large, S being a larger number. At very large δ/λ, $C_2 \simeq -1$ regardless of $\alpha(\delta/a)^{\frac{1}{2}}$, and hence (56) is a good approximation for all $\alpha(\delta/a)^{\frac{1}{2}}$. Also implicit in (56) is the assumption that

$$\left|\ln(k\chi S^{1-1/m})^{\frac{1}{2}}\frac{\lambda}{\delta}\right| \ll \tfrac{1}{2}k\chi S^{1-1/m}, \tag{58}$$

which is satisfied for values of δ/λ that range from very small to very large compared with unity.

Equations (47) and (56) give the charge-density distribution in the pipe. The current density is obtained from (55) as

$$\frac{I_\infty}{q_w \pi a^2 V} = R\frac{\tau_w}{\rho V^2}\frac{\lambda^2}{a^2}\left(1 - \frac{\delta/\lambda}{\sinh(\delta/\lambda)}\right) + \frac{\delta/\lambda}{\sinh(\delta/\lambda)}\frac{2\lambda^2}{a\delta}. \tag{59}$$

This expression is valid for large α, but otherwise arbitrary, conditions. For $\delta/\lambda \to \infty$, (59) reduces to Cooper's result (31). For $\delta/\lambda \to 0$, it reduces – the first term on the right then being small – to our version of Klinkenberg's and Kozsman & Gavis's result (39). Note, however, that in this limit of high α the charge density outside the diffusion layer is non-uniform when $\delta/\lambda \to 0$, not uniform as assumed by Klinkenberg and Koszman & Gavis.

Low-α limit. In the limit of small α the pipe radius is small compared with the Debye length based on the core turbulent diffusivity, and the charge density may therefore be expected to be uniformly distributed in the turbulent regions outside the diffusion sublayer. Noting that, if α is small, then $\alpha(\delta/a)^{\frac{1}{2}}$ and δ/λ are also small, we expand the terms in (51) and (52) for small α, small $\alpha(\delta/a)^{\frac{1}{2}}$ and small δ/λ, keep only leading terms, and obtain

$$\left.\begin{aligned} C_2 &= -\frac{5a/8\lambda}{1+5\delta a/8\lambda^2}, \\ C_3 = C_5 &= \frac{1}{1+5\delta a/8\lambda^2}, \quad C_4 \to 0. \end{aligned}\right\} \tag{60}$$

The charge density is indeed distributed uniformly in the turbulent regions outside the diffusion sublayer, at a value which is q_w when $\delta a/\lambda^2 \ll 1$ and less than q_w otherwise.

In the same approximation, the convection current is obtained as

$$\frac{I_\infty}{q_w \pi a^2 V} \simeq \frac{\frac{17}{16}}{1+5\delta a/8\lambda^2}. \tag{61}$$

The term in (55) involving $\tau_w/\rho V^2$ is negligible in this limit. Equation (61), unfortunately, shows some of the inaccuracies we have introduced into the problem by modelling the region $\delta < y < 0{\cdot}25a$ as being plane. For $\delta a/\lambda^2 \ll 1$, the charge density is everywhere q_w, and the right-hand side of the equation should clearly be unity, not $\frac{17}{16}$. The quantity $\frac{17}{16}$ simply represents the ratio of our model's flow area $\pi(0{\cdot}75a)^2 + 2\pi a(0{\cdot}25a)$ to the real flow area πa^2. The term $\frac{5}{8}$ in the denominator is also an approximation: the correct value should be $\frac{1}{2}$. The origin of the $\frac{5}{8}$ can be understood by considering the development, valid for $\delta a/\lambda^2 \ll 1$, that led to (39). In this development the charge density in the core (uniform, in our present case) is obtained by equating Dq_w/δ with σE_r at $r \simeq a$, and E_r at $r \simeq a$ is evaluated from

$$2\pi a \epsilon E_r \simeq \int_0^a q 2\pi r\, dr. \tag{62}$$

This gives the factor as $\frac{1}{2}$. In our present case, we treat the region $0 < y < \delta$ as plane, and compute E_r from

$$2\pi a \epsilon E_r \simeq \int_0^{0{\cdot}25} q\, dy + \int_0^{0{\cdot}75} q 2\pi r\, dr, \tag{63}$$

which gives rise to the $\frac{5}{8}$ instead of the correct $\frac{1}{2}$.

Recognizing the artifices introduced by our approximation, we write the correct solution for $\alpha \ll 1$ as

$$\frac{I_\infty}{q_w \pi a^2 V} = \frac{1}{1+\delta a/2\lambda^2}. \tag{64}$$

Our equation (61) overestimates the correct solution by 6 % at $\delta a/\lambda^2 \ll 1$, and underestimates it by 15 % at $\delta a/\lambda^2 \gg 1$. Note that similar difficulties did not arise in the limit $\alpha \gg 1$, where the charge was distributed near $r \simeq a$, and the one-dimensional approximation gave accurate results.

A general expression for charging current. When the operating point passes through the region where the value of α goes through unity, neither (59) nor (64) is, strictly speaking, supposed to be applicable. However, in this region $\delta a/\lambda^2$ is large compared with unity, and δ/λ is small (see figure 1), and (59) reduces to the same form as (64) (note that the first term in (59) is small when α is of order unity). This suggests that the following expression should accurately represent I_∞ for all values of α:

$$\frac{I_\infty}{q_w \pi a^2 V} = R\frac{\tau_w}{\rho V^2}\frac{\lambda^2}{a^2}\left(1-\frac{\delta/\lambda}{\sinh(\delta/\lambda)}\right)+\frac{\delta/\lambda}{\sinh(\delta/\lambda)}\frac{1}{1+\delta a/2\lambda^2}. \tag{65}$$

Equation (65) reduces to (59) in the limit of high α, and to (64) in the limit of low α. For α in the transition region near unity, it reduces to the form – independent of the specific value of α – shared by both (59) and (64).

Equation (65), then, approximates the solution for *all* turbulent-flow conditions and all fluid conductivities.

4. Correction for the effect of finite pipe length

The expression for charging current derived in §3 applies for 'infinite' pipes where both the velocity and charge distributions have attained asymptotic, unchanging forms. In this section we consider how the charge evolves in finite pipes, and how the asymptotic state of the previous analysis is attained.

We assume in what follows that the velocity profile is fully developed throughout most of the pipe. Since the development length for the velocity profile in turbulent flow is conservatively about 10^2 pipe radii (Schlichting 1979), this means that we restrict our consideration to pipes that are longer than several hundred radii.

When the uncharged fluid first enters the pipe, the charge density is everywhere zero, including in the region adjacent to the pipe wall. As a result, a current will pass between the fluid and the pipe (14), and the charge density in the fluid at the wall will begin to rise toward the equilibrium value q_w. If the exchange current in (14) is very large compared with the actual current at the wall, the charge density at the wall will be very close to q_w, regardless of the fact that current passes between the fluid and the wall. In what follows, we shall assume that this is the case. Thus, the boundary condition at the wall will be

$$\bar{q} \simeq q_w \quad \text{at} \quad r = a, \tag{66}$$

throughout the pipe, including the inlet region. The charge set up at the wall disperses into the body of the flow, fed by current from the wall. The evolution of the charge distribution in the turbulent flow is governed by (8).

The high-conductivity limit ($\lambda \ll \delta$). When the conductivity is so high that $\lambda \ll \delta$, the development length for the charge density is short compared with that of the velocity, and hence the infinite-pipe expression for charging current is always applicable as long as we restrict our consideration to pipes that are longer than several hundred radii.

That this is so can be demonstrated by estimating the charge-layer development length. In the limit $\lambda \ll \delta$ the charge is confined to a region λ near the wall. The development length is approximately the product of λ^2/D, the time it takes charge to diffuse to a distance λ from the wall, and $\tau_w \lambda/\mu$, the characteristic mean velocity in the charged layer inside the viscous sublayer. The charge-density development length L is thus of the order

$$\frac{L}{a} \sim \frac{\tau_w \lambda^3}{\mu a D}, \tag{67}$$

which can be rewritten as

$$\frac{L}{a} \sim \frac{1}{4}\frac{\tau_w}{\rho V^2} R^2 \left(\frac{\delta}{a}\right)^3 S \left(\frac{\lambda}{\delta}\right)^3$$
$$\sim 1{\cdot}6 \times 10^4 R^{-\frac{3}{8}} (\lambda/\delta)^3, \tag{68}$$

where we have used (17), (18) and (20) to get the last expression. Since $\lambda \ll \delta$, (68) shows that L/a is very small compared with unity, even at the lowest turbulent Reynolds numbers, L can safely be taken to be negligible if the pipe length is measured in hundreds of radii or more. The charge density is thus fully developed whenever $\lambda \ll \delta$.

The moderate and low-conductivity limit ($\lambda \gg \delta$). In this limit is is convenient to follow the integral approach used by Koszman & Gavis (1962*a*). We integrate (8) from $r = 0$ to $r = a$, and divide by πa^2:

$$\frac{2}{a^2}\frac{\partial}{\partial x}\int_0^a \bar{v}_x \bar{q} r\, dr - \frac{2D}{a}\left(\frac{\partial \bar{q}}{\partial r}\right)_{r=a} + \frac{2}{a^2}\frac{\sigma}{\epsilon}\int_0^a \bar{q} r\, dr = 0. \tag{69}$$

Next, we make two assumptions, to be verified later. First, that the total charge in the pipe is large compared with the charge contained in the diffusion sublayer, or

$$q_{av}\pi a^2 \gg q_w 2\pi a \delta, \tag{70}$$

where q_{av} is defined by (33). Secondly, that the charge gradient at the wall can be approximated by

$$\left(\frac{\partial q}{\partial r}\right)_{r=a} \simeq \frac{q_w - q_{av}}{\delta}. \tag{71}$$

If these two assumptions apply (more on that below), (69) simplifies to

$$V\frac{\partial q_{av}}{\partial x} + \frac{2D}{a}\frac{(q_w - q_{av})}{\delta} + \frac{\sigma}{\epsilon} q_{av} = 0. \tag{72}$$

The mechanism of charging is as follows. As the fluid enters the pipe, it encounters a non-equilibrium charge condition at the wall. A current from the wall results, and the charge density quickly (immediately, in our present model) rises to q_w at the wall, and diffuses from there across the diffusion sublayer into the turbulent core of the flow (the second term in (72)), causing a rate of increase (the first term) of average charge, most of which is in the core. The average charge in the core, however, sets up an electric field at the wall, and this in turn results in a migration current which tends to counteract the diffusion current and reduce the rate at which charge is fed to the core (this is the last term in (72)). At sufficiently large x the diffusion and migration currents exactly balance, and the final asymptotic state results.

Assuming that $q_{av} = 0$ at $x = 0$, the solution to (72) is

$$q_{av} = \frac{q_w}{1 + a\delta/2\lambda^2}(1 - e^{-x/L}), \tag{73}$$

where

$$L = \frac{\epsilon V/\sigma}{1 + 2\lambda^2/a\delta} \tag{74}$$

is a characteristic charge development distance. The convective-charging current $I = q_{\mathrm{av}} \pi a^2 V$ is thus given by

$$I = I_\infty(1 - e^{-x/L}), \tag{75}$$

where

$$I_\infty = \frac{q_{\mathrm{w}} \pi a^2 V}{1 + a\delta/2\lambda^2} \tag{76}$$

is the charging current for an infinite pipe in this limit of $\delta \ll \lambda$.

There remains the task of verifying that our simplifying assumptions (70) and (71) are consistent with our result. To get the most conservative test of (70) we substitute from (73) and (74) for q_{av} in the limits $x \ll L$ and $a\delta/\lambda^2 \gg 1$ (high conductivity) and find that (70) is equivalent to

$$\frac{x}{a} \gg \tfrac{1}{2} R\, S \left(\frac{\delta}{a}\right)^2. \tag{77}$$

Using (18) and (20) in the defining relation for δ, we have

$$\frac{\delta}{a} = 118 R^{-\frac{7}{8}} S^{-\frac{1}{3}}, \tag{78}$$

and hence our requirement reads

$$\frac{x}{a} \gg 7 \times 10^3 R^{-\frac{3}{4}} S^{\frac{1}{3}}. \tag{79}$$

This inequality is satisfied for typical values of R and S for pipes longer than about $10^3 a$, and it remains a fair approximation even at the lowest fully turbulent Reynolds numbers if the Schmidt number is 500, say.

The first requirement for the second simplifying assumption (71) is that the charge density be linear in y inside the diffusion sublayer. That this must indeed be so can be seen by examining the orders of magnitude of the various terms in (8) inside the sublayer, where $\bar{v}_x \sim \tau_{\mathrm{w}} \delta/\mu$ and $D_{\mathrm{T}}/D \ll 1$. Such an examination shows that the fractional change in $\partial\bar{q}/\partial y$ across the diffusion layer is equal to a term – arising from the last term in (8) – of order δ^2/λ^2, which is small in this case, plus a term – arising from the first term – of order

$$\frac{1}{4}\left(\frac{\delta}{a}\right)^3 \frac{\tau_{\mathrm{w}}}{\rho V^2} R^2 S \frac{a}{L}. \tag{80}$$

Inserting (79), (20) and (74), it becomes clear that this term, too, is very small at all typical turbulent-flow conditions. Hence the charge gradient is indeed constant inside the diffusion layer.

The second requirement for (71) is that the charge either be distributed at about the value q_{av} in the turbulent core outside the diffusion sublayer, or, if that is not the case, that $q_{\mathrm{av}} \ll q_{\mathrm{w}}$, so that the effect of q_{av} in (71) is negligible. This requirement is satisfied. For $\alpha < 1$, we have seen that the charge is distributed uniformly in the core (note that the infinite-pipe solution correctly indicates the *shape* – though not the level – of the charge-density profile in the turbulent core, since the development length for the tubulent dispersion of charge in the core is the same as that for the velocity profile, and hence small compared with the pipe lengths being considered). For $\alpha > 1$, the charge distribution is non-uniform, but $q_{\mathrm{av}} \ll q_{\mathrm{w}}$. Equation (71) thus remains a good approximation at all values of α.

The assumptions on which (74) is based are therefore justified. Note that (74) differs from the development length derived by Koszman & Gavis (1962*a*) through the term $2\lambda^2/a\delta$ in the denominator. This term is important only in the limit of extremely low conductivity, when $2\lambda^2/a\delta \gg 1$. In this limit the development length is not the electrical relaxation length $\epsilon V/\sigma$, as derived by Koszman & Gavis, but is completely independent of fluid conductivity, and controlled only by the diffusion sublayer.

An expression for the general case. Our analysis is restricted to pipes longer than several hundred radii, for which the velocity profile can be considered to be fully developed throughout most of the pipe. We have shown that, when $\lambda \ll \delta$, the charge density is always fully developed in such a pipe, and that, when $\lambda \gg \delta$, the effect of pipe length on the charging current is given by (74) and (75). Now, one can show quite straightforwardly, using the relation for δ and λ given in §2, that (74) and (75) predict a correction factor either equal to unity or very close to unity for all fully turbulent flow conditions if one extrapolates to λ of order δ and smaller than δ. Thus, (74) and (75) blend into the length effect (or absence thereof) for $\lambda \ll \delta$ before the approximations on which they are based become invalid. It follows that (75) describes the length effect accurately for *all* conditions, if I_∞ is taken as the *general* expression (65) for charging current for an infinite pipe.

5. Summary

We have shown that the convective current in a turbulent pipe flow can be expressed as

$$I = I_\infty(1-e^{-x/L}), \tag{81}$$

where

$$\frac{I_\infty}{q_w \pi a^2 V} = R\frac{\tau_w}{\rho V^2}\frac{\lambda^2}{a^2}\left(1-\frac{\delta/\lambda}{\sinh(\delta/\lambda)}\right)+\frac{\delta/\lambda}{\sinh(\delta/\lambda)}\frac{1}{1+a\delta/2\lambda^2}, \tag{82}$$

$$L = \frac{\epsilon V/\sigma}{1+2\lambda^2/a\delta}. \tag{83}$$

The fluid's conductivity enters these equations through the numerator in L and through the Debye length

$$\lambda \equiv (\epsilon D/\sigma)^{\frac{1}{2}}. \tag{84}$$

The turbulent-flow conditions are defined through the Reynolds number $R = \rho V 2a/\mu$, the shear stress τ_w at the wall, and the diffusion-sublayer thickness δ. We suggest a diffusion-sublayer thickness consistent with the high-Schmidt-number mass transfer data of Petukhov (1970):

$$\frac{\delta}{a} = \frac{11\cdot 7}{S^{\frac{1}{3}}}\frac{\nu}{a(\tau_w/\rho)^{\frac{1}{2}}}. \tag{85}$$

For the shear stress one can use Blasius's simple correlation $\tau_w/\rho V^2 = 0\cdot 0396 R^{-\frac{1}{4}}$, or, if better accuracy is required for $R > 10^5$, for example Prandtl's implicit expression (Schlichting 1979).

To apply our equation one needs to know not only the usual flow variables, but the diffusion coefficient(s) of the ionic species and, most importantly, the equilibrium charge density q_w in the fluid at the pipe wall. This latter is an interfacial property of the wall/fluid combination which can in principle be measured.

The results obtained here are based on several key assumptions.

(i) The velocity profile is assumed to be fully developed throughout the pipe. This restricts the solution to pipes that are several hundred radii – a figure of 10^3 radii can be taken as conservative – or more in length.

(ii) The charge density is assumed to be everywhere small, in the sense of (2). Since the charge density is nowhere greater than q_w, this requires that $|q_w| < Fc_0$.

(iii) The solution for charge development in a finite pipe assumes that the actual current at the wall is small compared with the exchange current that characterizes the charge/current relation at the wall [see (14)], so that $q \simeq q_w$ at the wall even if the current is nonzero. There being no information available on typical charge/current relations at boundaries of liquid hydrocarbons, little can be said about the reasonableness of this assumption at this time.

Some simplifying assumptions were also made in the governing equations. The one which probably has the most important effect on our solution is our oversimplified representation (19) of the mean axial velocity profile. This will introduce some inaccuracy into our solution when $\lambda \sim \delta$, but not when $\lambda \ll \delta$ or $\lambda \gg \delta$. Some inaccuracy is, however, a not-unexpected price for a completely explicit, analytic solution that covers all turbulent flow conditions.

A comparison of the present theory with available experimental data will be presented in a later paper. Abedian (1979) has already shown that with some exceptions, for which reasons can be found, the present theory is in essential agreement with available well-documented data, including the very -high-Debye-length data of Hampel & Luther (1957), which did not agree with Klinkenberg's (1959, 1964) and Koszman & Gavis's (1962*a*) theories. These comparisons are based, however, on certain assumptions about the wall charge density q_w, of which no direct measurements have ever been made in liquid hydrocarbons: one must assume that q_w is a property of the fluid/wall combination, and is independent of flow conditions, but scales in direct proportion to the ion concentration c_0.

This research was supported by the National Science Foundation under Grants ENG-7620283 and 7826547.

REFERENCES

ABEDIAN, B. 1979 Ph.D. thesis, Massachusetts Institute of Technology.

BOUMANS, A. A. 1957 *Physica* **23**, 1007–1055.

COOPER, W. F. 1953 *Brit. J. Appl. Phys.* **4** (*Suppl.* **2**), S11–S15.

DEISSLER, R. G. 1955 *NACA Tech. Rep.* no. 1210.

GIBSON, N. & LLOYD, F. C. 1970*a* *Chem. Engng Sci.* **25**, 87–95.

GIBSON, N. & LLOYD, F. C. 1970*b* *J. Phys.* D: *Appl. Phys.* **3**, 563–573.

GOODFELLOW, H. D. & GRAYDON, W. F. 1968 *Chem. Engng Sci.* **23**, 1267–1280.

HAMPEL, B. & LUTHER, H. 1957 *Chemie-Ing.-Tech.* **29**, 323–329.

HELMHOLTZ, H. 1879 *Ann. Physik und Chemie* **243**, 337–382.

HUBER, P. W. & SONIN, A. A. 1977 *J. Colloid Interface Sci.* **61**, 109–125.

KLINKENBERG, A. 1959 *Genie Chimique* **82**, 149–157.

KLINKENBERG, A. 1964 *Chemie-Ing.-Tech.* **36**, 283–290.

KLINKENBERG, A. & VAN DER MINNE, J. L. 1958 *Electrostatics in the Petroleum Industry*. Elsevier.

KOSZMAN, I. & GAVIS, J. 1962*a* *Chem. Engng Sci.* **17**, 1013–1022.

KOSZMAN, I. & GAVIS, J. 1962*b* *Chem. Engng Sci.* **17**, 1023–1040.

LAUFER, J. 1954 *NACA Rep.* no. 1174.

PETUKHOV, B. S. 1970 In *Advances in Heat Transfer* (ed. J. P. Hartnett & T. F. Irvine), vol. 6. Academic.

PRIBYLOV, V. N. & CHERNYI, L. T. 1979 *Fluid Dynamics* **14**, 844–849.

QUARMBY, A. & ANAND, R. K. 1969 *J. Fluid Mech.* **38**, 433–455.

REICHARDT, H. 1951 *Arch. Ges. Wärmetechnik* **6/7**, 129–143.

SCHLICHTING, H. 1979 *Boundary Layer Theory*, 7th edn. McGraw-Hill.

SCHWING, R. C. 1970 *J. Colloid Interface Sci.* **32**, 432–443.

SPALDING, D. B. 1961 *J. Appl. Mech.* **28**, 455–457.

TOUCHARD, G. 1978 *J. Electrostatics* **5**, 463–476.